Directions: Show all of your work and clearly indicate your answers. Correct answers with no work may not receive credit.

1. Calculate the iterated integrals
   a. \[ \int_1^2 \int_0^2 (y + 2xe^y) \, dx \, dy \]
      \[ = \int_1^2 (2y + 4e^y) \, dy = 3 + 4e^2 - 4e \]
   b. \[ \int_0^1 \int_0^x \cos(x^2) \, dy \, dx \]
      \[ = \int_0^1 \cos(x^2) \, x \, dx = \left[ \frac{1}{2} \sin(x^2) \right]_0^1 = \frac{1}{2} \sin(1) \]
   c. \[ \int_0^1 \int_0^{\sqrt{1-y^2}} y \sin(x) \, dz \, dx \]
      \[ = \int_0^\pi \int_0^1 \sin(x) y \sqrt{1-y^2} \, dy \, dx = \int_0^\pi \sin(x) \, dx \cdot \int_0^1 y \sqrt{1-y^2} \, dy \]
      \[ = \left[ -\cos(x) \right]_0^\pi \cdot \left[ -\frac{1}{2} \right] \left[ 2 \left( 1-y^2 \right)^{3/2} \right]_0^1 = 2 \]

2. Evaluate \[ \int \int_D \, xy \, dA \], where \( D = \{(x,y) : 0 \leq y \leq 1, \, y^2 \leq x \leq y + 2 \} \).
   \[ = \int_0^1 \int_{y^2}^{y+2} xy \, dx \, dy = \int_0^1 \frac{1}{2} \left( y^3 + 4y^2 + 4y - y^5 \right) \, dy = \frac{41}{24} \]

3. Evaluate \[ \int \int_D \frac{y}{1+x^2} \, dA \], where \( D \) is bounded by \( y = \sqrt{x}, \, y = 0, \, x = 1 \).
   \[ = \int_0^1 \int_0^{\sqrt{x}} \frac{y}{1+x^2} \, dy \, dx = \frac{1}{2} \int_0^1 \frac{x}{1+x^2} \, dx = \frac{1}{4} \ln(1+x^2) \bigg|_0^1 = \frac{1}{4} \ln(2) \]
4. Calculate the iterated integral: \( \int_0^1 \int_x^1 \cos(y^2) \, dy \, dx \). [Change order of integration.]

\[
= \int_0^1 \int_0^1 \cos(y^2) \, dx \, dy = \int_0^1 \cos(y^2) \cdot y \, dy = \frac{1}{2} \sin(y^2) \bigg|_0^1 = \frac{1}{2} \sin(1)
\]

5. Calculate the iterated integral: \( \int_0^1 \int_0^{\sqrt{y}} \frac{ye^{x^2}}{x^3} \, dx \, dy \). [Change order of integration.]

\[
= \int_0^1 \int_0^x \frac{ye^{x^2}}{x^3} \, dy \, dx = \int_0^1 \left( \frac{e^{x^2}}{x} \right) \bigg|_0^1 - \frac{1}{2} \, dx = \int_0^1 \frac{e^{x^2}}{x} \cdot \frac{x^4}{2} \, dx
\]

\[
= \frac{1}{2} \int_0^1 e^{x^2} x \, dx = \frac{1}{4} e^{x^2} \bigg|_0^1 = \frac{1}{4}(e - 1)
\]

6. Find the volume of the solid under the paraboloid \( z = x^2 + 4y^2 \) and above the rectangle \( R = [0, 2] \times [1, 4] \).

\[
= \int_1^4 \int_0^2 (x^2 + 4y^2) \, dx \, dy = \int_1^4 \left( \frac{8}{3} + 8y^2 \right) \, dy = \frac{528}{3} = 176
\]

7. Calculate the volume of the solid under the surface \( z = x^2 y \) that lies above the triangle in the \( xy \)-plane with vertices \( (1,0), (2,1) \) and \( (4,0) \). [Warning: tedious algebra ahead.]

\[
= \int_0^{2y+4} x^2 y \, dx \, dy = \frac{1}{3} \int_0^1 \left( -9y^4 + 45y^3 - 99y^2 + 63y \right) \, dy = \frac{53}{20}
\]

Alternatively,

\[
= \int_1^2 \int_0^{x-1} x^2 y \, dx \, dy + \int_2^4 \int_0^{\frac{x}{2}+2} x^2 y \, dx \, dy = \frac{31}{60} + \frac{32}{15} = \frac{53}{20}
\]

8. Evaluate \( \int \int_D \sqrt{x^2 + y^2} \, dA \), where \( D \) is the region in the first quadrant bounded by the circle \( x^2 + y^2 = 2 \) and the \( x- \) and the \( y- \)axes.

\[
= \int_0^\pi \int_0^{\sqrt{2}} r \cdot r \, dr \, d\theta = \frac{r^3}{3} \bigg|_0^{\sqrt{2}} \cdot \theta \bigg|_0^\pi = \frac{2\sqrt{2}}{3} \cdot \frac{\pi}{2} = \frac{\pi \sqrt{2}}{3}
\]

9. Evaluate \( \int \int_D x \, dA \), where \( D \) is the region above the \( x \)-axis bounded by the circle \( x^2 + y^2 = 9 \).

\[
= \int_0^\pi \int_0^3 r \cos(\theta) \cdot r \, dr \, d\theta = \frac{r^3}{3} \bigg|_0^3 \cdot \sin(\theta) \bigg|_0^\pi = 9 \cdot 0 = 0
\]
10. Calculate the volume of the solid bounded by the cylinder \( x^2 + y^2 = 4 \) and the planes \( z = 0 \) and \( y + z = 3 \).

\[
\begin{align*}
&= \int_0^{2\pi} \int_0^2 (3-r \sin(\theta)) \cdot r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 (3r - r^2 \sin(\theta)) \, dr \, d\theta = \int_0^{2\pi} \left( 6 - \frac{8}{3} \sin(\theta) \right) \, d\theta \\
&= \left( \frac{12\pi}{3} \right) - \left( 0 + \frac{8}{3} \right) = 12\pi
\end{align*}
\]

11. Evaluate \( \iiint_E xy \, dV \), where \( E = \{(x, y, z) : 0 \leq x \leq 3, 0 \leq y \leq x, 0 \leq z \leq x+y\} \).

\[
\begin{align*}
&= \int_0^3 \int_0^x \int_0^{x+y} xy \, dz \, dy \, dx = \int_0^3 \int_0^x (x^2y + xy^2) \, dy \, dx = \int_0^3 \frac{5}{6} x^4 \, dx = \frac{1}{6} x^5 \bigg|_0^3 = \frac{81}{2}
\end{align*}
\]

12. Evaluate \( \iiint_E z \, dV \), where \( E \) is the solid region bounded by the planes \( z = 0 \), \( z = 1 + x + y \), and the cylinder \( x^2 + y^2 = 1 \) in the first octant. [Warning: highly involved and technical algebra here.]

\[
\begin{align*}
&= \int \int \int_D \int_0^{1+x+y} z \cdot dA = \int_0^2 \int_0^1 \int_0^{1+r(\cos(\theta)+\sin(\theta))} z \cdot r \, dr \, d\theta \\
&= \int_0^2 \int_0^1 \left( 1 + r(\cos(\theta) + \sin(\theta)) \right)^2 \cdot r \, dr \, d\theta \\
&= \frac{1}{2} \int_0^2 \int_0^1 \left( r + 2r^2(\cos(\theta) + \sin(\theta)) + r^3(\cos(\theta) + \sin(\theta))^2 \right) \, dr \, d\theta \\
&= \frac{1}{2} \int_0^2 \left( \frac{1}{2} + \frac{2}{3}(\cos(\theta) + \sin(\theta)) + \frac{1}{4}(1 + \cos(\theta) \sin(\theta)) \right) \, d\theta \\
&= \frac{1}{2} \left( \frac{1}{2} \theta + \frac{2}{3}(\sin(\theta) - \cos(\theta)) + \frac{1}{4}(\theta + \frac{1}{2} \sin^2(\theta)) \right) \bigg|_0^\frac{\pi}{4} = \frac{3\pi}{16} + \frac{19}{24}
\end{align*}
\]

13. Find the volume of the solid inside the cylinder \( x^2 + y^2 = 4 \) under the paraboloid \( z = x^2 + y^2 \) and above the plane \( z = 0 \).

\[
\begin{align*}
&= \int \int \int_D \int_0^{x^2+y^2} 1 \, dz \, dA = \int_0^{2\pi} \int_0^2 \int_0^{r^2} 1 \cdot r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^2 r^2 \, dr \, d\theta \\
&= \frac{r^4}{4} \bigg|_0^2 \cdot \theta \bigg|_0^{2\pi} = 8\pi
\end{align*}
\]
14. Evaluate \( \iiint_E (x^2 + y^2) \, dV \), where \( E \) is the solid region above the \( xy \)-plane between the spheres \( x^2 + y^2 + z^2 = 1 \) and \( x^2 + y^2 + z^2 = 4 \).

\[
= \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 \left( \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta \right) \cdot \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\
= \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 \rho^4 \sin^3 \phi \, d\rho \, d\theta \, d\phi \\
= \frac{\rho^5}{5} \left|_1^2 \right| \cdot \theta \left|_0^{2\pi} \right| \cdot \int_0^{\pi/2} \sin^3 \phi \, d\phi = \frac{31}{5} \cdot \frac{2\pi}{3} = \frac{124\pi}{15}
\]

To evaluate the last integral, do the following:

\[
\int_0^{\pi/2} \sin^3 \phi \, d\phi = \int_0^{\pi/2} (1 - \cos^2 \phi) \sin \phi \, d\phi = \int_0^{\pi/2} \left( \sin \phi - \cos^2 \phi \sin \phi \right) \, d\phi \\
= \left( -\cos \phi + \frac{1}{3} \cos^3 \phi \right) \bigg|_0^{\pi/2} = \frac{2}{3}
\]

15. Evaluate \( \iiint_E y^2 \sqrt{x^2 + y^2 + z^2} \, dV \), where \( E \) is the solid region in the first octant inside the sphere \( x^2 + y^2 + z^2 = 1 \).

\[
= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \left( \rho^2 \sin^2 \phi \sin \theta \cdot \rho \right) \cdot \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\
= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^5 \sin^3 \phi \sin \theta \, d\rho \, d\theta \, d\phi \\
= \int_0^1 \rho^5 \, d\rho \cdot \int_0^{\pi/2} \sin^2 \theta \, d\theta \cdot \int_0^{\pi/2} \sin^3 \phi \, d\phi \\
= \left[ \frac{1}{6} \rho^6 \right]_0^1 \cdot \left[ \frac{1}{2} \theta - \frac{1}{4} \sin(2\theta) \right]_0^{\pi/2} \cdot \left[ -\cos \phi + \frac{1}{3} \cos^3 \phi \right]_0^{\pi/2} \\
= \frac{1}{6} \cdot \frac{\pi}{4} \cdot \frac{2}{3} = \frac{\pi}{36}
\]

To evaluate the third integral, see above. To do the second, use the half-angle identity:

\[
\int_0^{\pi/2} \sin^2 \theta \, d\theta = \int_0^{\pi/2} \frac{1}{2} \left( 1 - \cos(2\theta) \right) \, d\theta = \frac{1}{2} \left( \theta - \frac{1}{2} \sin(2\theta) \right) \bigg|_0^{\pi/2} = \frac{\pi}{4}
\]
16. Find the area of the cardioid $r = 4 + 3 \cos(\theta)$. 

![Cardioid Graph]

\[
\text{Area} = \iint_D 1 \, dA = \int_0^{2\pi} \int_0^{4 + 3 \cos \theta} r \, dr \, d\theta = \int_0^{2\pi} \frac{1}{2} (4 + 3 \cos \theta)^2 \, d\theta \\
= \frac{1}{2} \int_0^{2\pi} \left( 16 + 24 \cos \theta + 9 \cdot \frac{1 + \cos 2\theta}{2} \right) \, d\theta = \frac{41 \pi}{2}.
\]

17. Find the area of one petal of the polar rose $r = \cos(3\theta)$.

For this problem, it is important to get your bounds right. It is easy to see that $0 \leq r \leq \cos(3\theta)$. To find the bounds for $\theta$, we look at the graph of the curve.

![Polar Rose Graph]

We want only one petal, so we need to find an interval of $\theta$ values where $r$ starts at zero and then goes back to zero. That is, we need to find the angles $\theta$ where $r = 0$ and our range will be the interval between two consecutive such $\theta$ values.

Consider $\cos(t)$. We know that $\cos(t) = 0$ when $t = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots$. In our case, $t = 3\theta$, so we have $\theta = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \cdots$. The petal on the right is the one where $\theta$ goes from $-\frac{\pi}{6}$ to $\frac{\pi}{6}$, so we will use this interval for $\theta$. (You can use any consecutive $\theta$ values, however.) Therefore,

\[
\text{Area} = \iint_D 1 \, dA = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^{\cos(3\theta)} r \, dr \, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2} (\cos 3\theta)^2 \, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{4} (1 + \cos 6\theta) \, d\theta = \frac{\pi}{12}.
\]
18. Find the volume of the solid region bounded below by the cone \( z = \sqrt{x^2 + y^2} \) and above by the sphere \( x^2 + y^2 + z^2 = 1 \). (This region is sometimes called an ice cream cone.)

\[
\text{Volume} = \iiint_E 1 \, dV = \int_0^{\pi/4} \int_0^{2\pi} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\
= \left( \frac{\rho^3}{3} \right)_0^1 \cdot (\theta)_0^{2\pi} \cdot (-\cos \phi)|_0^{\pi/4} = \frac{2\pi}{3} \left( -\frac{\sqrt{2}}{2} + 1 \right)
\]